The Solow Model

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Introduction to Economic Growth

Production

We assume that real GDP is produced according to

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} \tag{1}$$

where

- K_t is the stock of physical capital (e.g. buildings, equipment)
- L_t is the number of workers/people
- A_t is the level of productivity; how efficiently we use capital and labor
- α tells us how important capital is relative to labor

Constant returns

This production function has **constant returns to scale**. If you double the rival inputs K_t and L_t, output doubles (A_t is non-rival, discussed later),

$$(zK_t)^\alpha(A_t zL_t)^{1-\alpha} = z^\alpha z^{1-\alpha} K_t^\alpha (A_t L_t)^{1-\alpha} = zY \tag{2}$$

so we have constant returns because $\alpha + (1-\alpha) = 1$.

GDP per capita

GDP per capita is defined by $y_t = Y_t/L_t$. We can write this like:

$$
\begin{aligned}
y_t &= \frac{Y_t}{L_t} \\
&= \frac{K_t^{\alpha}(A_t L_t)^{1-\alpha}}{L_t} \\
&= \left(\frac{K_t}{A_t L_t}\right)^{\alpha} A_t. \qquad (3)
\end{aligned}
$$

- K_t/A_tL_t is sometimes called “capital per efficiency unit”. The rate of return on capital will depend on this ratio, and along a BGP this ratio will end up constant.
- This means that the “extra” productivity term A_t will drive growth in GDP per capita.

The growth rate

Take logs and derivatives of y_t. Start with logs:

$$\begin{aligned} \log y_t &= \alpha(\log K_t/A_tL_t) + \log A_t \\ &= \alpha(\log K_t - \log A_t - \log L_t) + \log A_t. \end{aligned} \tag{4}$$

and then take derivative with respect to time

$$g_y = \alpha(g_K - g_A - g_L) + g_A. \tag{5}$$

- The term in parentheses represents transitory growth driven by accumulation of capital; this generates slow transitions.
- The productivity growth term g_A remains and drives growth along the BGP.

Wages and the return to capital

This connects to the other parts of the BGP. Assume that a large number of competitive firms in the economy produce output using the prior function, and try to maximize profits

$$\pi_t = Y_t - w_t L_t - r_t K_t.$$

where w_t is the wage and r_t is the return to capital. Their first-order conditions (e.g. wage equals marginal product) are

$$\begin{aligned} w_t &= \frac{\partial Y_t}{\partial L_t} = (1-\alpha)\frac{Y_t}{L_t} \\ r_t &= \frac{\partial Y_t}{\partial K_t} = \alpha\frac{Y_t}{K_t}. \end{aligned}$$

Labor's share of GDP

The first-order conditions imply

$$\frac{w_t L_t}{Y_t} = 1 - \alpha.$$

The production function is designed to ensure that labor's share of GDP is constant, to match the BGP facts.

The return to capital

From the first-order condition the return to capital is

$$r_t = \frac{\alpha Y_t}{K_t}. \tag{6}$$

Along a BGP r_t is constant, so it must be that Y_t/K_t is constant. What's that ratio?

$$\frac{Y_t}{K_t} = \frac{K_t^\alpha (A_t L_t)^{1-\alpha}}{K_t} = \left(\frac{A_t L_t}{K_t}\right)^{1-\alpha}.$$

This is what tells us that the ratio K/AL must be constant along a BGP; it ensures that r_t is constant along a BGP.

Labor and Productivity

Two of the items in the production function are assumed to just grow exogenously at a given rate. Later we'll look at what determines these growth rates.

Labor:

$$L_t = L_0 e^{g_L t} \tag{7}$$

Productivity:

$$A_t = A_0 e^{g_A t} \tag{8}$$

Capital accumulation

Solow's model relies on one additional assumption regarding how capital accumulates:

$$\begin{aligned} dK &= I_t - \delta K_t \\ &= s_I Y_t - \delta K_t \end{aligned} \tag{9}$$

- dK is the change in the capital stock (implicitly per unit of time)
- I_t is gross capital formation
- s_I is the fraction of GDP used for gross capital formation
- δ is the deprecation rate, the fraction of capital that breaks down each period

The growth rate of capital

Use the accumulation and divide it by K_t

$$g_K = s_I \frac{Y_t}{K_t} - \delta.$$

where $g_K = dK/K_t$ is the growth rate of capital.

We know $Y_t/K_t = (A_t L_t/K_t)^{1-\alpha}$ from before so

$$\begin{aligned} g_K &= s_I \frac{K_t^{\alpha}(A_t L_t)^{1-\alpha}}{K_t} - \delta \\ &= s_I \left(\frac{A_t L_t}{K_t}\right)^{1-\alpha} - \delta. \end{aligned} \tag{10}$$

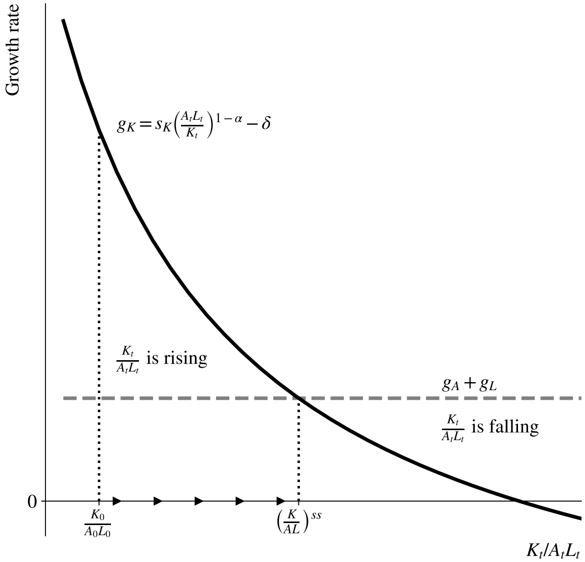
The growth rate of capital

$$g_K = s_I \left(\frac{A_t L_t}{K_t} \right)^{1-\alpha} - \delta. \tag{11}$$

The growth rate of capital depends:

- *negatively* on the ratio $K_t/A_t L_t$. The bigger the stock of K relative to AL, the slower the growth rate.
- *positively* on s_I. The more resources we commit to building capital, the faster it grows.
- *negatively* on δ. The faster capital breaks down, the slower it grows.

The dynamics of capital



Steady state

The dynamics ensure that the ratio evolves towards a central point where $g_K = g_A + g_L$. That point is a *steady state* for K/AL. Solve for that ratio:

$$g_A + g_L = s_I \left(\frac{AL}{K}\right)^{1-\alpha} - \delta.$$

which yields

$$\left(\frac{K}{AL}\right)^{ss} = \left(\frac{s_I}{g_A + g_L + \delta}\right)^{\frac{1}{1-\alpha}}. \tag{12}$$

Steady state

$$\left(\frac{K}{AL}\right)^{ss} = \left(\frac{s_I}{g_A + g_L + \delta}\right)^{\frac{1}{1-\alpha}}. \tag{13}$$

While the ratio K/AL is constant at the steady state, the *size* of the ratio in that steady state is

- Higher when s_I is large. If we commit more resources to capital accumulation, the capital stock will be relatively large in steady state.
- Lower when g_L is large. If the population grows quickly, it is hard for capital to “keep up” and the ratio is lower in steady state.
- Lower when g_A is large. If productivity grows quickly, it is also hard for capital to “keep up”. Don’t get confused, this doesn’t mean productivity growth is bad for the economy.

Steady state growth

Remember that no matter what, the growth rate of GDP per capita is determined by

$$g_y = \alpha(g_K - g_A - g_L) + g_A.$$

THe dynamics of K/AL lead to steady state where $g_K = g_A + g_L$, so

$$g_y^{ss} = g_A. \tag{14}$$

The source of long-run growth

In the long run the growth rate of GDP per capita is determined *only* by the growth rate of productivity, g_A.

Balanced growth path

The economy ends up at a steady state. Is this steady state consistent with a balanced growth path?

- The growth rate of GDP per capita is constant, $g_y^{BGP} = g_A$. ✓
- Labor's share of GDP is constant $wL/Y = 1 - \alpha$. ✓
- The share of GDP used for capital accumulation is s_I. ✓
- The real interest rate, r_t, is constant. ??

Real interest rate

What's r? We know that

$$r_t = \frac{\alpha Y_t}{K_t}. \tag{15}$$

and

$$\frac{Y_t}{K_t} = \frac{K_t^\alpha (A_t L_t)^{1-\alpha}}{K_t} = \left(\frac{A_t L_t}{K_t}\right)^{1-\alpha}.$$

so in steady state Y/K is constant and

$$r^{ss} = \alpha \frac{g_A + g_L + \delta}{s_I}. \tag{16}$$

r is constant in steady state, consistent with BGP. ✓

Solow and BGP

The key elements of the Solow model are

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$$

and

$$g_K = s_I \frac{Y_t}{K_t} - \delta.$$

which leads to:

Solow and BGP

The dynamics of capital accumulation ensure that the economy ends up in steady state, and in that steady state the economy is on a BGP.

Other growth rates

K/AL is constant in steady state, but other important things are growing:

- Total GDP. $g_Y = g_y + g_L$ so $g_Y^{ss} = g_A + g_L$
- Total capital. $g_K^{ss} = g_A + g_L$
- Consumption per capita, $c = (1 - s_I)y$ so $g_c^{ss} = g_A$
- Capital per capita, $k = K/L$, so $g_k^{ss} = g_A$

The level of GDP per capita

The BGP is definitive about growth rates, but not the level of GDP per capita. At all times we have

$$y_t = \left(\frac{K_t}{A_t L_t}\right)^{\alpha} A_t.$$

so in steady state

$$y_t^{BGP} = \left(\frac{s_I}{g_A + g_L + \delta}\right)^{\frac{\alpha}{1-\alpha}} A_t, \tag{17}$$

and note that this still grows over time due to A_t growing.

The level of GDP per capita

We tend to think in terms of *log* of GDP per capita in figures,

$$\begin{aligned}\log y_t^{BGP} &= \frac{\alpha}{1-\alpha}\log\left(\frac{s_I}{g_A+g_L+\delta}\right)+\log A_t \\ &= \frac{\alpha}{1-\alpha}\log\left(\frac{s_I}{g_A+g_L+\delta}\right)+\log A_0+g_A t.\end{aligned}$$

given $\log A_t = \log A_0 + g_A t$.

Note that this is the equation of a *line*, with $\log y_t^{BGP}$ as the "y-variable" and t as the "x-variable".

The level of GDP per capita

The intercept and slope of this line:

$$\log y_t^{BGP} = \underbrace{\left(\frac{\alpha}{1-\alpha}\log\left(\frac{s_I}{g_A+g_L+\delta}\right)+\log A_0\right)}_{\text{Intercept}} + \underbrace{g_A}_{\text{Slope}}\, t.$$

The intercept determines the level of GDP per capita along the BGP. Note that:

- If s_I is higher, the level of GDP p.c. is higher, even though the growth rate (slope) is not.
- If the initial level of productivity, A_0, is higher, GDP p.c. is higher, even though the growth rate (slope) is not.

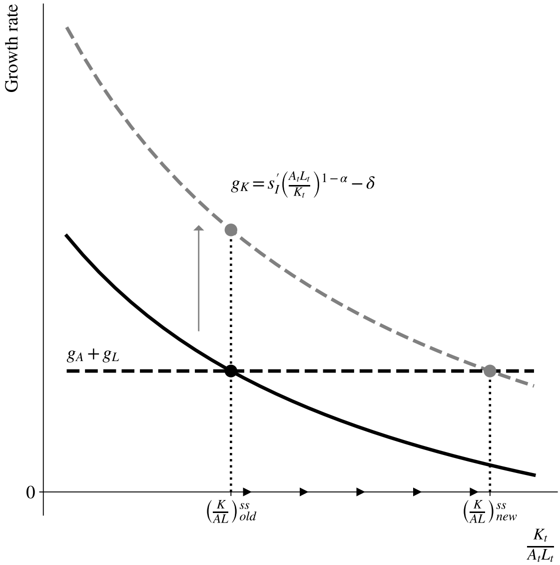
Changes in the economy

What happens if a parameter like s_I changes? This *moves* the BGP and the economy slowly adjusts until it reaches the new BGP. To understand work through distinct questions:

- What happens to the dynamics of K/AL immediately after the change?
- What happens to K/AL in the long run (steady state)?
- What happens to the level of the BGP in response to the change?
- What do the K/AL dynamics imply about how the economy reaches the BGP?

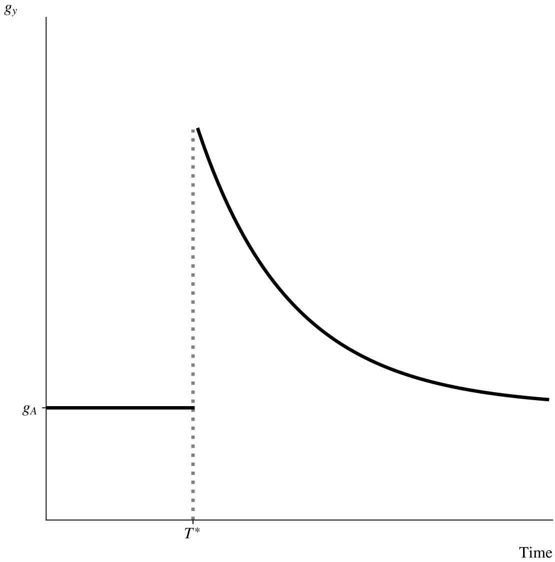
Dynamics of K/AL

If s_I icnreases, g_K shifts up *immediately*, and the steady state is larger in the long run.



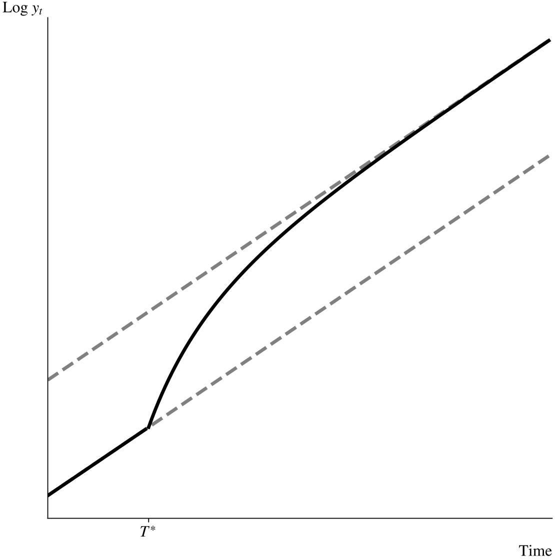
The growth rate

Because g_K goes up immediately, g_y goes up immediately. In the long run, g_y goes back to g_A.



The level of GDP per capita

The increase in s_I shifts the BGP *up*. g_y implies a slow transition towards the new BGP.



Transitory growth

The increase in s_I is an example of **transitory growth**.

- g_y is above g_A for a while, but eventually $g_y \Rightarrow g_A$
- Transitory growth occurs as an economy moves towards steady state
- This growth is transitory because the dynamics ensure that $g_K \Rightarrow g_A + g_L$
- Differences in growth rates across countries tend to be transitory

$$g_y = \underbrace{\alpha(g_K - g_A - g_L)}_{\text{Transitory}} + \underbrace{g_A}_{\text{Long-run}}$$